Super-Nyquist Rateless Coding for Intersymbol Interference Channels

Uri Erez
Tel Aviv University, Ramat Aviv, Israel
Email: uri@eng.tau.ac.il

Gregory W. Wornell
Dept. EECS, MIT, Cambridge, MA
Email: gww@mit.edu

***Abstract*— A rateless transmission architecture is developed for communication over Gaussian intersymbol interference channels, based on the concept of super-Nyquist (SNQ) signaling. In such systems, the signaling rate is chosen significantly higher than the Nyquist rate of the system. We show that such signaling, when used in conjunction with good "off-the-shelf" base codes, simple linear redundancy, and minimum mean-square error decision feedback equalization, results in capacity-approaching, low-complexity rateless codes for the time-varying intersymbol-interference channel. Constructions for both single-input / single-output (SISO) and multi-input / multi-output (MIMO) ISI channels are developed.**

I. Introduction

In traditional digital communication, achieving high throughput when the channel state allows is accomplished by selecting high-order signal constellations. However, an alternative approach, originally proposed several decades ago [1], exploits super-Nyquist (SNQ) (equivalently, faster-than-Nyquist) signaling. In SNQ signaling, the symbols are taken from a fixed constellation, typically BPSK or QPSK, independent of the transmission rate. Higher rates are achieved by increasing the signaling rate—i.e., the rate at which the symbols are modulated onto the bandlimited pulse shape—beyond the Nyquist rate. Thus, in SNQ systems, the signaling rate is decoupled from the transmission bandwidth, and can greatly exceed the transmission bandwidth.

Because SNQ modulation introduces ISI, it necessitates the use of equalization, which traditionally made it unappealing for early applications; see, e.g., [2]. In this paper, however, we establish that SNQ signaling has some particularly valuable properties for communication over Gaussian intersymbol interference (ISI) channels where the transmitter knows neither the channel impulse response nor the maximal rate that my be supported by the channel. In particular, we establish the somewhat surprising result that the use of SNQ signaling allows for highly efficient joint design of the physical and link layers. Indeed, from such signaling we develop a rich family of low-complexity, capacity-approaching rateless codes for scalar ISI channels, which have natural extensions to vector ones.

This work was supported in part by the ONR under MURI Grant No. N00014-07-1-0738, by AFOSR under Grant No. FA9550-11-1-0183, and by the Israel Science Foundation under Grant No. 1557/12.

II. System and Channel Model

We consider a linear dispersive Gaussian channel for which the complex baseband channel output takes the form

$$y(t) = h(t) * x(t) + z(t),$$

where $z(t)$ is additive white Gaussian noise (AWGN) with one-sided power spectral density N_0, and where $x(t)$ is the input, which is subject to a power constraint $\mathbb{E}\{|x(t)|^2\} \leq P$ and bandwidth constraint W. The associated white-input capacity of the channel is

$$C_{[\mathrm{b/s}]} = \int_{-W/2}^{W/2} \log\left(1 + \frac{P|H(f)|^2}{N_0 W}\right) \mathrm{d}f. \qquad (1)$$

We consider pulse-amplitude modulation, whereby

$$x(t) = \sum_n s[n]\, g(t - nT), \qquad (2)$$

where T is the symbol duration and $T_0 = 1/W$ is the Nyquist sampling time. The associated "over-signaling" ratio is thus $L = T_0/T$. After matched filtering and sampling at the symbol rate, the equivalent discrete-time channel is

$$y[n] = k[n] * s[n] + z[n], \qquad (3)$$

where $k[n] = k(nT)$, where $k(t) = h^*(-t) * h(t) * g(-t)^* * g(t)$, and where

$$\begin{aligned} S_{zz}(e^{j2\pi f}) &= \frac{N_0}{2} K(e^{j2\pi f}) \\ &= \frac{N_0}{2T} \sum_i |H(f/T + i/T)|^2 |G(f/T + i/T)|^2. \end{aligned}$$

The pulse shape is required to be limited to system bandwidth W, i.e., $g(t)$ satisfies $G(f) = 0$ for $|f| > W/2$. To simplify our development, we largely restrict our attention to the case

$$g(t) = \mathrm{sinc}(t/T_0), \text{ with } \mathrm{sinc}(u) \triangleq \sin(\pi u)/(\pi u). \qquad (4)$$

Taking the symbols $s[n]$ to be independent identically-distributed (i.i.d.) circularly symmetric complex Gaussian with power P/L results in a (proper) Gaussian random input signal

$x(t)$ with power P. It follows that the capacity of the discrete-time channel (3) is

$$
\begin{aligned}
&C_{[\mathrm{b/SNQ\ symbol}]} \\
&\quad = \int_{-1/2}^{1/2} \log\left(1 + \frac{(P/L)\cdot K(e^{j2\pi f})}{N_0}\right) \mathrm{d}f \\
&\quad = T\int_{-1/2T}^{1/2T} \log\left(1 + \frac{P\sum_i |H(f+i/T)|^2 |G(f+i/T)|^2}{T_0 N_0}\right) \mathrm{d}f \\
&\quad = T\int_{-1/2T}^{1/2T} \log\left(1 + \frac{P|H(f)|^2|G(f)|^2}{T_0 N_0}\right) \mathrm{d}f, \qquad (5)
\end{aligned}
$$

where the last equality follows from the fact that $g(t)$ is bandlimited.

Note that for sinc modulation (4), $x(t)$ has a flat power spectrum over the bandwidth W, and the modulation achieves the white-input capacity (1) for any L, i.e., (5) specializes to

$$
\begin{aligned}
C_{[\mathrm{b/SNQ\ symbol}]} &= \frac{1}{LW}\int_{-W/2}^{W/2} \log\left(1 + \frac{P|H(f)|^2}{N_0 W}\right) \mathrm{d}f \\
&= \frac{T}{T_0}\frac{1}{W} C_{[\mathrm{b/s}]}. \qquad (6)
\end{aligned}
$$

III. Linear SNQ Rateless Coding

Consider now packetized transmission where the packet size is large but otherwise plays no role in the analysis. We consider a simplified model where the channel response experienced throughout transmission of the mth packet, $m = 1, \dots, M$, is linear time-invariant (LTI) but the impulse response, which we denote by $h_m(t)$, may vary from packet to packet. The channel input-output relation for the transmission of the mth packet is therefore

$$
y_m[n] = s_m[n] * k_m[n] + z_m[n], \qquad (7)
$$

where $k_m[n] = k_m(nT)$ and $k_m(t) = h_m^*(-t) * h_m(t) * g^*(-t) * g(t)$. Assuming discrete-time white-input transmission for all packets, it follows from (6), that the mutual information (in b/SNQ symbol) corresponding to each packet is

$$
\begin{aligned}
C_{m[\mathrm{b/SNQ\ symbol}]} &= \int_{-1/2}^{1/2} \log\left(1 + \frac{PK_m(e^{2\pi f})}{N_0 L}\right) \mathrm{d}f \\
&= \frac{1}{L}\frac{1}{W}\int_{-W/2}^{W/2} \log\left(1 + \frac{P|H_m(f)|^2}{N_0 W}\right) \mathrm{d}f,
\end{aligned}
$$

where the second equality holds for ideal sinc modulation. Upon receiving a set $\mathcal{S} \subset \{1, \dots, M\}$ of packets, the aggregate mutual information is thus

$$
C(\mathcal{S}) = \sum_{m\in\mathcal{S}} C_m. \qquad (8)
$$

Our aim is to design a low complexity coding and modulation scheme that (simultaneously) approaches $C(\mathcal{S})$ for all sets $\mathcal{S} \subset \{1, \dots, M\}$ without requiring the transmitter to have knowledge of the capacities C_m. Rather, for any given chosen target rate R, and no knowledge of the channel, transmission should be successful whenever $C(\mathcal{S}) > R$ holds for the received set of packets $\mathcal{S}$.

We proceed to describe the proposed linear rateless SNQ construction. All the signals $s_m[n]$ are obtained from a single coded stream $s[n]$ according to

$$
s_m[n] = v_m[n]\, s[n], \qquad (9)
$$

where $v_m[n]$ are sequences to be specified. The transmitted signal corresponding to packet m is thus

$$
x_m(t) = \sum_n s[i]\, v_m[i]\, g(t - iT).
$$

Provided we choose the sequences $v_m[n]$ so that the transmitted signals $x_m(t)$ are statistically independent and the $s_m[n]$ are white circularly-symmetric complex Gaussian processes, the mutual information corresponding to each packet remains C_m and furthermore the aggregate mutual information from the receipt of multiple packets is the sum of the individual ones.

A simple means to achieve this is by taking $v_m[n] = e^{-j2\pi mn/L}$. Accordingly, we define

$$
x_m[n] \triangleq x_m(nT) = \sum_i s[i]\, e^{-j2\pi im/L}\, g[n-i], \qquad (10)
$$

where $g[n] = g(nT) = \mathrm{sinc}(n/L)$, with $g(t)$ as in (4). Using (10), we see that the transmit signals "shifted-back in frequency" in this case are

$$
\tilde{x}_m[n] \triangleq e^{j2\pi mn/L}\, x_m[n] = \sum_i s[i]\, g_m[n-i],
$$

where $g_m[n] = e^{j2\pi mn/L}\, g[n]$.

Clearly, requiring that the signals $\{x_m(t)\}$ be mutually independent is equivalent to requiring that the associated discrete-time signals $\{x_m[n]\}$ be. Furthermore, the latter holds if and only if $\{\tilde{x}_m[n]\}$ are mutually independent. Therefore, it suffices to verify the last condition. Since the signals $\tilde{x}_m[n]$ are jointly Gaussian and stationary, they are independent if their cross-spectra

$$
\begin{aligned}
&S_{\tilde{x}_{m_1}\tilde{x}_{m_2}}(e^{j2\pi f}) \\
&\quad = S_{ss}(e^{j2\pi f})\, G_{m_1}(e^{j2\pi f})\, G_{m_1}^*(e^{j2\pi f}) \\
&\quad = \frac{1}{T^2} S_{ss}(e^{j2\pi f})\, G\left(\frac{f + m_1/L}{T} \bmod \frac{1}{T}\right) \\
&\qquad\qquad \cdot G^*\left(\frac{f + m_2/L}{T} \bmod \frac{1}{T}\right)
\end{aligned}
$$

vanish. Since $G(f/T)$ occupies no more than $1/L$ of the SNQ frequency band, it follows that there is no overlap between the frequency responses $G\left((f + m/L)/T \bmod 1/T\right)$ for different values of m, and hence $S_{x_{m_1}x_{m_2}}(e^{j2\pi f})$ indeed vanishes for $m_1 \neq m_2$.

A. Receiver Architecture

A low-complexity receiver architecture suffices to approach the associated information-theoretic limits. In particular, specializing (7) to the case (9) with our choice $v_m[n] = e^{-j2\pi mn/L}$, the equivalent "shifted back" channel model is

$$
\tilde{y}_m[n] \triangleq y_m[n]\, e^{j2\pi mn/L} = \tilde{k}_m[n] * s[n] + \tilde{z}_m[n], \qquad (11)
$$

where $\tilde{k}_m[n] = k_m[n]\, e^{j2\pi mn/L}$. Since these channels do not overlap in frequency, they may be added without loss, resulting in the effective scalar ISI channel

$$\begin{aligned}\tilde{y}[n] &= \sum_{m\in\mathcal{S}} y_m[n]\, e^{j2\pi mn/L} \\ &= s[n] * \left(\sum_{m\in\mathcal{S}} \tilde{k}_m[n]\right) + \sum_{m\in\mathcal{S}} \tilde{z}_m[n].\end{aligned}$$

For such ISI channels, the unbiased MMSE decision-feedback equalizer (DFE) is an information-lossless receiver structure [3], [4]. In particular, to approach capacity, one can use Guess-Vanarasi interleaving [3] and a single (fixed-rate) base code designed for an AWGN channel. In essence, every symbol is replaced by a different codeword and thus the DFE decision device acts on codewords rather than symbols.

IV. MIMO-SNQ: Extending SNQ to MIMO systems

A straightforward extension of the preceding architecture to a multi-input multi-output (MIMO) system, which we term MIMO-SNQ, is as follows. The particular channel model of interest is

$$\mathbf{y}_m(t) = \mathbf{H}_m(t) * \mathbf{x}_m(t) + \mathbf{z}_m(t),$$

where there are N_t transmit and N_r receive elements. In turn, with input of the form $\mathbf{x}_m(t) = \sum_n \mathbf{x}_m[n]\, g(t - nT)$, the associated discrete-time channel, after applying a matrix matched filter, is

$$\mathbf{y}_m[n] = \mathbf{K}_m[n] * \mathbf{x}_m[n] + \mathbf{z}_m[n],$$

where $\mathbf{K}_m[n] = \mathbf{K}_m(nT)$ with $\mathbf{K}_m(t) = \mathbf{H}_m^\dagger(-t) * \mathbf{H}_m(t) * g^*(-t) * g(t)$.

We employ a single stream transmission architecture based on the application of time-varying DFT beamforming to a scalar signal $s[n]$. Specifically, we assume that the transmitted signal corresponding to packet m is formed as $\mathbf{x}_m[n] = s[n]\,\mathbf{v}_m[n]$, where $\mathbf{v}_m[n] = \mathbf{v}[n]\, e^{-j2\pi mn/L}$ with

$$\mathbf{v}[n] = \begin{bmatrix}1 & e^{-j2\pi n/N_\mathrm{t}} & \cdots & e^{-j2\pi(N_\mathrm{t}-1)n/N_\mathrm{t})}\end{bmatrix}^\mathrm{T}.$$

The effective received signal for packet m is, after frequency shifting [cf. (11)],

$$\tilde{\mathbf{y}}_m[n] = \mathbf{y}_m[n]\, e^{j2\pi mn/L} = \sum_l \tilde{\mathbf{K}}_m[l]\,\mathbf{v}[n-l]\, s[n-l] + \tilde{\mathbf{z}}_m[n], \tag{12}$$

where $\tilde{\mathbf{K}}_m[n] = \mathbf{K}_m[n]\, e^{j2\pi mn/L}$. Note that the effective channel (12) is a periodically varying MIMO-ISI channel with period N_t. We refer to each of the N_t induced substreams as "phases." As in the SISO case, we may employ a DFE at the receiver, but due to the time-varying nature of the effective channel, for each phase a different set of N_r feedforward filters is applied to the channel output vector sequence. Thus, the equalizer is also periodic with period N_t.

Note the covariance matrix of the transmitted vector for our SNQ modulation is white. We conclude that for this modulation, the transmitted signal is white over all degrees of freedom as long as the oversignaling rate satisfies $L \geq N_\mathrm{t} M$. However, this doesn't guarantee capacity can be achieved. In particular, we associate with each of the N_t "phases" a signal-to-interference-plus-noise ratio (SINR) value corresponding to the associated DFE slicer input. Equivalently, we may associate with each such phase a corresponding capacity. Hence, while the sum of the per-phase capacities equals the white-input capacity of the MIMO channel, the per-phase capacities are in general not equal. Moreover, since the variation is unknown to the transmitter, in a Guess-Varanasi transmission architecture, a fixed code rate is used, and thus the achievable rate is determined by the minimum of the per-phase capacities.

It is worth emphasizing that this SINR variation across phases is analogous to the SINR variation across streams in a V-BLAST system, in which independently coded streams are sent over the antennas [5]. For this reason, V-BLAST serves as a useful benchmark with which to compare the performance of SNQ modulation.

A. Parallel channels

In some cases, MIMO-SNQ is strictly capacity achieving. For example, consider the special case of N parallel ISI channels. This model is essentially equivalent to using SNQ modulation for transmission over a block-varying ISI channel as considered in Section III, with the N parallel channels replacing the L consecutive blocks of the SISO ISI channel. As we have shown that SNQ modulation is an optimal scheme in such a scenario, it follows that MIMO-SNQ is optimal for the case of parallel channels.

B. Channels Without Temporal ISI

As another class of channels of interest, consider the special case in which there is no temporal ISI and only inter-channel interference (ICI) is present, i.e., $\mathbf{H}_m(t) = \mathbf{H}_m$.

While for diagonal $\mathbf{H}_m$ MIMO-SNQ is capacity achieving, there exist other $\mathbf{H}_m$ for which MIMO-SNQ achieves zero rate. For example, if $\mathbf{H}_m$ is the (rank-one) matrix of all 1's, there will exist an SNQ Nyquist substream that experiences a zero-capacity channel since a vector of all 1's is orthogonal to $\mathbf{v}[n]$ for $n \neq lN_\mathrm{t}$, all l. Hence, MIMO-SNQ achieves zero rate, while V-BLAST achieves a strictly positive rate. However, for all but such pathological $\mathbf{H}_m$, MIMO-SNQ supports a rate that grows with SNR. By contrast, it is well know that for all rank-one channel matrices V-BLAST performance is interference-limited, i.e., is bounded with increasing SNR.

More generally, when the channel matrix $\mathbf{H}_m$ is drawn from a random ensemble, its performance is on average never worse that V-BLAST, and has significant advantages, particularly when keeping in mind that in V-BLAST an ordering is forced in the detection process, while MIMO-SNQ requires no such ordering since the scheme is inherently more symmetric.

We consider the average throughput for an ensemble of $\mathbf{H}_m$ with i.i.d. circularly symmetric complex Gaussian entries. The resulting average throughput of MIMO-SNQ is depicted in Fig. 1, along with that for both fixed- and optimized-decoding-order V-BLAST As the plot reflects, the performance

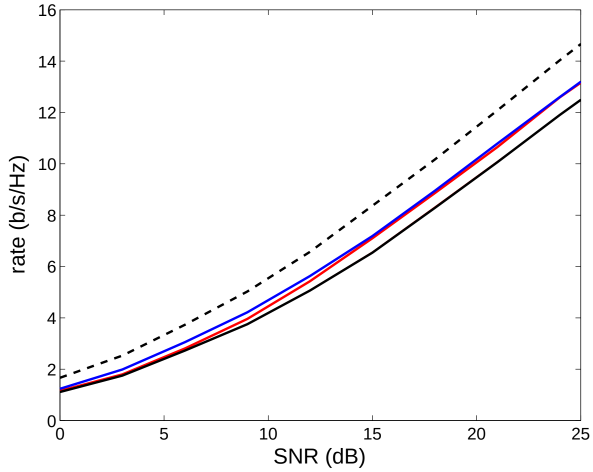

Fig. 1. Average MIMO-SNQ performance over random 2×2 MIMO channels without temporal ISI. The successively higher solid curves correspond to V-BLAST with fixed-order decoding, MIMO-SNQ, and V-BLAST with optimum-order decoding, and the dashed curve indicates capacity.

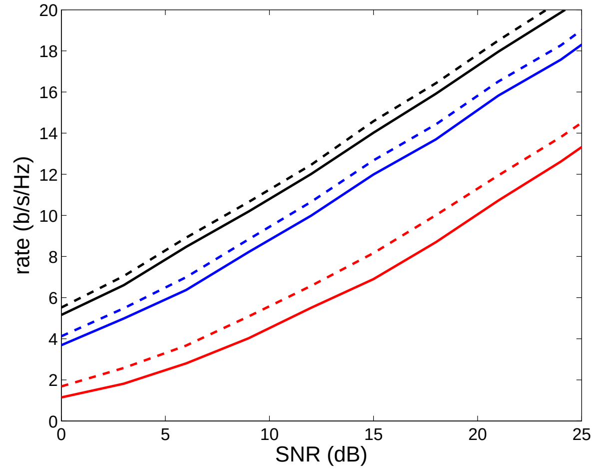

Fig. 2. Average MIMO-SNQ performance over random 2×2 MIMO channel with spatio-temporal uncorrelated Gaussian scattering. The successively higher solid curves correspond to 1, 5, and 10 taps of ISI, respectively, and the associated dashed curves indicate the corresponding capacities.

of MIMO-SNQ modulation lies in between the two and approaches the latter at high SNR.

We can also relate MIMO-SNQ performance to that of D-BLAST [6]. In particular, as is well known, D-BLAST can achieve capacity, but to do so requires a base code designed for time-varying channels. In that sense, MIMO-SNQ can also achieve capacity provided practical such base codes exist. However, when we require that a communication architecture has the property that the base code sees an AWGN channel, Fig. 1 reflects that MIMO-SNQ can perform as well as V-BLAST with an optimized decoding order. In constrast, when the same constraint is imposed on D-BLAST, the result is V-BLAST with a fixed decoding order.

C. Spatio-Temporal ISI Channels

When there is also temporal ISI, MIMO-SNQ is even more attractive, as we next illustrate. We now consider a random MIMO channel model where the Nyquist-rate equivalent discrete-time matrix channel impulse response $\mathbf{K}_m[n]$ is of finite length and each Nyquist-rate tap is drawn i.i.d. over spatial and time dimensions according to a circularly-symmetric complex Gaussian distribution. Fig. 2 depicts the expected (averaged over the ensemble) capacity of MIMO-SNQ for different channel lengths. We observe that the gap-to-capacity decreases as the channel length grows confirming that SNQ modulation is able to exploit the temporal diversity afforded by the channel.

Moreover, such behavior is not specific to such i.i.d. ensembles. Indeed, Table I describes the performance of MIMO-SNQ for a typical 2×2 underwater acoustic communication channel realization from the recent KAM-11 experiment.[1] Performance was numerically evaluated for an oversignaling rate of $L = 2$. In this case, the equivalent discrete-time (Nyquist-rate) baseband channel impulse responses are 100 taps long. As the table reflects, MIMO-SNQ is effectively capacity achieving for this channel, with a gap to capacity of less than 0.5 dB.

TABLE I

MIMO-SNQ SPECTRAL EFFICIENCY (B/S/HZ) ON A SAMPLE 2×2 UNDERWATER ACOUSTIC CHANNEL OF LENGTH 100 TAPS

	SNR (dB)							
	0	2	4	6	8	10	12	14
Capacity	0.95	1.31	1.76	2.30	2.94	3.68	4.51	5.41
SNQ	0.86	1.20	1.63	2.16	2.80	3.53	4.37	5.29

[1]The authors thank Qing He for providing this sample channel.

REFERENCES

[1] J. E. Mazo, "Faster-than-Nyquist signaling," *Bell Sys. Techn. J.*, vol. 54, pp. 1451–1462, Oct. 1975.

[2] G. J. Foschini, "Contrasting performance of faster binary signaling with QAM," *Bell Sys. Techn. J.*, vol. 63, pp. 1419–1445, Oct. 1984.

[3] T. Guess and M. K. Varanasi, "An information-theoretic framework for deriving canonical decision-feedback receivers in Gaussian channels," *IEEE Trans. Inform. Theory*, vol. 51, pp. 173–187, Jan. 2005.

[4] G. D. Forney, Jr., "Shannon meets Wiener II: On MMSE estimation in successive decoding schemes.," in *Proc. Allerton Conf. Commun., Contr., Computing*, (Monticello, IL), Oct. 2004.

[5] P. W. Wolniansky, G. J. Foschini, G. D. Golden, and R. A. Valenzuela, "V-BLAST: An architecture for realizing very high data rates over the rich-scattering wireless channel," in *Proc. URSI Int. Symp. Signals, Systems, Electronics*, (Pisa, Italy), pp. 295–300, Oct. 1998.

[6] G. Foschini, R. Fischer, J. Huber, and G. Kamp, "Layered space-time architecture for wireless communication in a fading environment when using multiple antennas," *Bell Sys. Techn. J.*, pp. 41–46, Autumn 1996.